

# Chaos in Periodically Perturbed Monopole + Quadrupole Like Potentials.

*P. S. Letelier<sup>1</sup> and W. M. Vieira<sup>2</sup>*

Departamento de Matemática Aplicada-IMECC  
Universidade Estadual de Campinas  
13081-970 Campinas, S.P., Brazil

The motion of a particle that suffers the influence of simple inner (outer) periodic perturbations when it evolves around a center of attraction modeled by an inverse square law plus a quadrupole-like term is studied. The equations of motion are used to reduce the Melnikov method to the study of simple graphics.

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<sup>1</sup>e-mail: letelier@ime.unicamp.br

<sup>2</sup>e-mail: vieira@ime.unicamp.br

## 1. Introduction.

Since the pioneering work of Poincaré [1] in celestial mechanics wherein the mathematical basis of deterministic chaos was established, the study of homoclinic phenomena has been the focus of increasing attention not only in celestial mechanics but in many different branches of physics, chemistry and biology [2]. Due to its universality and intrinsic mathematical interest, models in which unstable periodic orbits (UPOs) are subjected to small periodic perturbations has been one of the main paradigms of deterministic chaos [3]. An analytical tool to study these models is the Melnikov function that describes the transversal distance between the unstable and stable manifolds emanating from an UPO. Its isolated odd zeros indicate the crossing of these manifolds, hence the onset of chaos [4]. The use of Melnikov method in connection to Kolmogorov-Arnold-Moser (KAM) theory for a large class of perturbations was studied by Holmes and Marsden [5], see also [6].

Examples of applications of the Melnikov method in gravitation are: the motion of a nearly symmetric heavy top [7], the motion of particles in a two-dimensional Stäckel potential separable with  $\cos m\varphi$  perturbations ( $m = 1, 2, \dots$ ) and other perturbations [8], extensions of the former work for perturbation of the three-dimensional Stäckel potentials [9] the gravitational collapse of cosmological models [10], the study of orbits around a black hole perturbed by either gravitational radiation [11, 12] or an external quadrupolar shell [13]

In this paper we consider the equatorial motion of a particle moving in a potential modeled by an inverse square law plus a quadrupole-like term. This potential describes more realistically the gravitational force of attraction of stars (like the sun) on planets and galaxy bulges that to a certain degree of approximation can be represented by a monopole plus a quadrupole on stars outside the galaxy. Also, this potential arises naturally in General Relativity in the study of test particles in a spherically symmetric attraction center (Schwarzschild black hole), the “quadrupole” being associated to the test particle angular momentum, see for instance [14].

Next, we consider a class of periodic perturbations of this integrable planar motion. The perturbations can be the result of periodic motion of mass distributions either inside or outside the orbit of interest. The fact that we were able to integrate in a closed form the equation of motion of the UPO is used to reduce the analysis of the Melnikov function to the study of simple graphics. We find that the perturbations will originate chaos in a variety of situations and range of parameters. Finally, we argue that chaotic behavior is generic for such a perturbations.

## 2. The homoclinic orbit.

We shall consider the orbit of a particle of mass  $m$  moving on a plane under the influence of a force described by a potential with a monopolar term plus a quadrupole-like contribution not necessarily small,

$$V = -km/R - Qm/R^3, \quad (1)$$

where  $k = GM$ ,  $M$  is the mass of the attraction center, and  $Q$  is the quadrupolar strength. In astronomy,  $Q = GMJ_2R_0^2/2$ , with  $R_0$  being a linear measure of the size of the central body and  $J_2$  the distortion parameter, see for instance [14]. All the undefined symbols have their usual meaning. Note that for a body with reflection symmetry the octupolar term is null ( $J_3 = 0$ ).

In General Relativity,  $Q$  is proportional to the square of the angular momentum of the test particle [14] .

The Lagrangian of the particle is

$$L = \frac{m}{2} \left[ \left( \frac{dR}{dt} \right)^2 + R^2 \left( \frac{d\phi}{dt} \right)^2 + \frac{2k}{R} + \frac{2Q}{R^3} \right]. \quad (2)$$

Since  $\phi$  is a cyclic variable we have the conservation law  $h = R^2 d\phi/dt$ . Hence, the motion of the particle is determined by the effective Hamiltonian

$$H_0 = \frac{P^2}{2m} + V_{eff}, \quad (P = m dR/dt), \quad (3)$$

$$V_{eff} = m \left( \frac{h^2}{2R^2} - \frac{k}{R} - \frac{Q}{R^3} \right). \quad (4)$$

The stationary values of  $V_{eff}$  are  $R = \frac{h^2}{2k}(1 \pm \sqrt{1-12\beta})$  with  $\beta = kQ/h^4$ . The minus (plus) sign correspond to the relative maximum (minimum). Note that  $\beta$  is a non dimensional parameter. We shall denote by  $R_{un}$  the UPO position given by the stationary value with minus sign (the maximum); we find that  $V_{eff}(R_{un}) = \frac{mh^2}{6R_{un}^2}(-1 + 2\sqrt{1-\beta})$ . Since, for large  $R$ , the potential  $V_{eff} \sim -\frac{k}{R} \leq 0$ , we need  $V_{eff}(R_{un}) \leq 0$  to have a periodic motion of the particle limited by  $R_{un}$  and  $R_m$ , the last being the turning point. Hence the parameter  $\beta$  is limited by  $1/16 = 0.0625 \leq \beta \leq 1/12 \sim 0.0833$ .

It is convenient to work with dimensionless quantities. We shall redefine our variables using the constants of the problem in the following way:  $r = kR/h^2$ ,  $\tau = k^2t/h^3$ ,  $\varepsilon = h^2E/mk^2$ , where  $E = H_0$  is the total energy of the particle. We recall that these quantities are proportional, respectively, to the radius, the period, and the energy of a particle of mass  $m$  moving in a circular orbit with areal velocity  $h$  around an attraction center characterized by  $k$ . The energy integral can be written as

$$2\varepsilon = \dot{r}^2 + 2v_{eff}, \quad (5)$$

$$2v_{eff} = 1/r^2 - 2/r - 2\beta/r^3, \quad (6)$$

where the overdot indicates derivation with respect to the time parameter  $\tau$ . In these new units,  $R_{un}$  is written as  $r_{un} = (1 - \sqrt{1-12\beta})/2$ . For  $\beta$  in the interval  $1/16 \leq \beta \leq 1/12$  we have  $1/4 \leq r_{un} \leq 1/2$ . In Fig. 1 we present a graphic of the potential (6) for the values of  $\beta = 0.068$  (top curve), 0.072, and 0.078 (bottom curve), the maximum value is located at  $r_{un}$ . A particle with energy  $\epsilon = v_{eff}(r_{un})$  will describe either the UPO itself ( $r = r_{un}$ ,  $\dot{r} = 0$ ) or the homoclinic orbit tending to the UPO as  $\tau \rightarrow \pm\infty$ . This homoclinic orbit is enclosed by circles of radii  $r_{un}$  and  $r_m = 2r_{un}(1 - r_{un})/(4r_{un} - 1)$ . Now, by making  $\epsilon = v_{eff}(r_{un})$  we can write (5) in the equivalent form

$$\frac{r^{3/2}\dot{r}}{(r - r_{un})\sqrt{r_m - r}} = \pm w_\beta, \quad (7)$$

with

$$w_\beta = \sqrt{(4r_{un} - 1)/(3r_{un}^2)}. \quad (8)$$

The differential equation (7) admits the quadrature

$$\mp w_\beta \tau = \sqrt{r(r_m - r)} + (r_m + 2r_{un}) \arctan \sqrt{\frac{r_m - r}{r}} + \frac{2r_m^{3/2}}{\sqrt{r_m - r_{un}}} \operatorname{arctanh} \sqrt{\frac{(r_m - r)r_{un}}{(r_m - r_{un})r}}. \quad (9)$$

We have chosen the constant of integration to have  $\tau = 0$  at  $r = r_m$ . A graphic  $\tau(r)$  of the positive branch of (9) is pictured in Fig. 2 for values of the parameter  $\beta = 0.064$  (top curve), 0.065, 0.066, 0.067, and 0.068 (bottom curve). We see that the particle takes a finite time to travel from  $r_m$  to the vicinity of  $r_{un}$  and then an infinite time to arrive (depart) to (from)  $r_{un}$  itself, wherein is located the UPO.

### 3. The Melnikov method.

Let us consider an integrable Hamiltonian  $H_0 = \frac{p^2}{2m} + V(q)$ , which admits at least one UPO with the corresponding homoclinic orbit, and also a small periodic perturbation described by the Hamiltonian function  $\eta H_1(q, p, t)$ . Then the transverse distance in phase space between the unstable and the stable manifolds emanating from the UPO is proportional to

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\} dt, \quad (10)$$

where the integral is taken along the unperturbed homoclinic orbit,  $t_0$  is an arbitrary initial time, and  $\{f, g\}$  are the usual Poisson brackets, see for instance [6]. If there is an intersection for some  $t_0$ , i.e., an isolated odd zero of  $M(t_0)$ , then there will be one for every  $t_0$ . This infinitely many crossings of manifolds will produce a tangle that is the signature of homoclinic chaos [4, 6].

The unperturbed system is Eqs. (5) and (6) with  $H_0 = \varepsilon$ , and  $(q, p) = (r, \dot{r})$ . For our purposes, it is enough to consider periodic perturbations of the particular form  $H_1 = f(r) \cos(\Omega\tau)$ . Perturbations of this form are representative of a very general situation: take the potential of a mass distribution

moving periodically and make a Fourier expansion in time and a multipolar expansion in the space variables. Next, consider approximate axial and reflection symmetry. Therefore, the generic term of the series expansion in the plane of the UPO ( $\theta = \pi/2$ ) will look like  $H_1$  above with  $f(r)$  a positive (negative) powers of  $r$  for mass distributions outside (inside) the homoclinic orbit. We find for the Melnikov function (10),

$$M(\tau_0) = \int_{-\infty}^{\infty} \frac{dr}{d\tau} \frac{df(r)}{dr} \cos[\Omega(\tau - \tau_0)] d\tau. \quad (11)$$

In our case, for the homoclinic orbit (9), we get

$$M(\tau_0) = -2K(\Omega) \sin(\Omega\tau_0), \quad (12)$$

$$K(\Omega) = \int_{r_{un}}^{r_m} \frac{df(r)}{dr} \sin[\Omega\tau(r)] dr, \quad (13)$$

where in the integrand  $\tau(r)$  means the positive branch of (9). Note that in (12) does not appear a term proportional to  $\cos(\Omega\tau_0)$ , its coefficient being null due to the fact that the homoclinic orbit has reflection symmetry with respect to the  $r$ -axis, and  $\cos(\Omega\tau)$  is an even function in the  $\tau$  variable. Thus the Melnikov function will have the required zeros as long as  $K(\Omega) \neq 0$ . The coordinate change  $\tau \rightarrow r$ , instead of the usual  $\tau \rightarrow \phi$ , allows us to pass from an infinite interval to a finite one in (13). This observation will play a key role in the evaluation of the function  $K(\Omega)$ .

#### 4. Particular cases.

Firstly, we shall consider the function  $K(\Omega)$  for exterior perturbations of the type  $f(r) = r^n$  with  $n = 1, 2, \dots$ . For a particle moving around the attraction center, they model periodic perturbations due to mass distributions beyond the orbit of the particle.  $n = 1$  represents dipolar contributions,  $n = 2$  quadrupolar, etc. Static quadrupolar and octopolar exterior contributions, inspired in the well known Hénon-Heiles model, give rise also to chaotic behavior in General Relativity [15].

Now, we shall study the integrand of  $K(\Omega)$  formed by the product of an oscillating function by a polynomial. Near  $r = r_{un}$  this oscillation is a rapid

one. To better understand this behavior we display in Fig. 3 a graphic of  $\sin[\Omega\tau(r)]$  for  $\beta = 0.064$  and  $\Omega = 0.05$  (top curve), 0.06, 0.07 and 0.08 (bottom curve). For  $\Omega > 0.08$  we will have more zeros in the interval  $1 < r < 10$  than for  $\Omega = 0.08$ . For  $\Omega < 0.05$  the curves will look like the one for  $\Omega = 0.05$ . Then it is easy to see that the integral  $K(\Omega)$  for  $f(r) = r$  will be non zero for  $\beta = 0.064$  and  $\Omega < 0.06$ . For  $n > 1$  we will have a more favorable situation. In Fig. 4 we plot the integrand of  $K(\Omega)$  for  $\beta=0.064$ ,  $\Omega=0.06$ , and  $f'(r) \equiv df/dr = r$  (top curve),  $r^2/10$ , and  $r^3/100$  (bottom curve), i.e., for a quadrupolar, octopolar and  $16^{th}$ -polar external periodic perturbations, respectively. In all cases the area under the curve is clearly not null, then we will have chaotic motion in these situations. The values of  $\Omega=0.06$  and  $0.07$  were obtained from  $\Omega_k \simeq \pi/\tau_k$ , where  $\tau_k$  corresponds to the maximum of the curvature ( $k = \tau''/(1 + (\tau')^2)^{3/2}$ ) of  $\tau(r)$  for the first value, and to the inflection point for the second one. From the previous analysis we see that the first value of  $\Omega$ , the one based on curvature, is better. Thus, we conclude that a perturbation with  $f(r) \sim r^\alpha$  with  $\alpha \geq 1$  and  $\Omega \leq \Omega_k$  will always be chaotic. A closer look shows that this criteria is good whenever  $r_m - r_{un} > 1$ , otherwise one needs  $\Omega \ll \Omega_k$ . It is also interesting to compare the perturbation frequency  $\Omega$  with  $\omega_{un} = 1/r_{un}^2$ , that is the angular frequency ( $\dot{\phi}$ ) of a particle at the UPO in dimensionless units. For  $\beta = 0.064$  we have  $\omega_{un}=15$ . So,  $\Omega \ll \omega_{un}$ .

Now we shall study perturbations on the motion of the particle that can be modeled by functions  $f(r) \sim r^{-n}$  with  $n=2, 3, \dots$ ;  $n=2$  represents monopolar contributions,  $n=3$  dipolar, etc. These type of perturbations can model forces due to distribution of masses with periodic motions that are placed inside the motion. Also, the case  $f(r) \sim \log r$ , which can be used to model the contribution of bodies with a spindly form, will be considered.

As in the previous case, we shall begin by analyzing a particular situation. In Fig. 5 we plot the integrand of  $K(\Omega)$  for  $\beta=0.068$ ,  $\Omega=0.15$ , and  $f'(r) = 1/(3r^2)$  (top curve),  $1/(9r^3)$ , and  $1/(27r^4)$ , (bottom curve). In this case we have a rapid oscillation (not shown in Fig. 5) near the point  $r_{un}$

(=0.285524) that produces very fine spines of sizes between  $[-4.0, 4.0]$  (top curve),  $[-4.5, 4.5]$ , and  $[-5.5, 5.5]$  (bottom curve). The areas under the positive parts of the curves are clearly greater than the areas under the negative ones, which is under very fine spines. So, also in this case  $K(\Omega)$  will be different from zero and the Melnikov method assures the occurrence of chaos. The value  $\Omega=0.15$  was obtained from  $\Omega_k \simeq \frac{\pi}{2\tau_k}$ . It is also instructive to compare the value of  $\Omega = 0.15$  with the UPO frequency. For  $\beta = 0.068$  we find  $\omega_{un} = 12$ . So, again  $\Omega \ll \omega_{un}$ . For powers  $n \gg 4$  we have that the relative size of the positive part of the area under the curve of the integrand of  $K(\Omega)$  begins to decrease and the area of the spines to increase. Hence, the graphic method is not appropriated in this case, though one can always numerically evaluate  $K(\Omega)$ . We will be back to this point later. A similar analysis shows that for  $f(r) \sim \log r$  we also have a strictly positive  $K(\Omega)$ . Therefore, we can say that for perturbations characterized by functions  $f'(r) \sim r^{-\alpha}$  with  $1 < \alpha < 4$  and  $\Omega < \Omega_k$ , we will have chaotic orbits. If one further restricts the range of  $\beta$  by the relation  $r_m - r_{un} < 2$ , one can have chaos with perturbations for any value of  $n > 1$  in  $f(r) \sim r^{-n}$ .

## 5. Discussion.

Let us first examine the domain of applicability of our results. The method is based in the existence of the homoclinic orbit that restrict the parameter  $\beta$  in a considerable way ( $1/16 \leq \beta \leq 1/12$ ). With the mass of the attraction center  $k = GM$  and the areal velocity of the test mass we construct our length scale  $k/h^2$ , which is kept arbitrary in our analysis.

We shall examine as a limit case the value of the parameters for particles moving near the minimum of the effective potential  $v_{eff}$  (cf. Fig. 1). In principle, these particles will feel to some extent the presence of the saddle point (in phase space), the maximum of the potential, that is the responsible for the instabilities that give rise to the chaotic motion. In particular, for a



circular orbit with radius  $r_P = R_P h^2/k$  at the minimum we get,

$$\beta = \frac{J_2 R_0^2 R_P^2}{2(R_P^2 + \frac{3}{2} J_2 R_0^2)^2}. \quad (14)$$

We have in non dimensional units that  $r_P = (1 + \sqrt{1 - 12\beta})/2$ . For the characteristic length of the central body we have  $r_0 = R_0 h^2/k = 2\beta/J_2$ . Since our potential (1) decays with the distance we need that  $r_P > r_0$ . So the parameter  $J_2$  that describes the oblateness of the central body plays a fundamental role in our analysis . From the limit condition  $R_0 = R_P$ , we get that  $\beta = 2J_2/(2 + 3J_2)^2$ . We recall that  $J_2 = 0$  for a spherical distribution of matter,  $1/4$  for a thin disk,  $1/2$  for a ring. To be more specific let us consider that all the matter is concentrated in a thin disk of radius  $R_0$ . We have in this case that  $0.0625 < \beta < 0.0661$ . It is instructive to make a table of the values of  $r_0$  and  $r_P$  and  $\varepsilon(r_0)$  and  $\varepsilon(r_P)$  for different values of  $\beta$ .

Table 1. Different parameters for  $J_2 = 1/4$  a disk like configuration.

$\beta$	$r_0(\beta)$	$r_P(\beta)$	$-\varepsilon(r_0)$	$-\varepsilon(r_P)$
0.0626	0.708	0.749	0.593	1.143
0.0630	0.710	0.747	0.594	1.142
0.0640	0.716	0.740	0.596	1.141
0.0650	0.721	0.735	0.599	1.137
0.0660	0.727	0.728	0.601	1.136

Therefore we have some interval of allowed energies and we can have chaos without a very fine tuning of the initial conditions. From the values in Table 1 we conclude that the potential well is rather deep. Furthermore the parameters also tells us that we can take for  $J_2$  a value smaller but close to  $1/4$ . In this case we can add some structure to the disk: some thickness and/or a small central bulge.

In this work we have presented a graphic method of implementing the classic Melnikov method for significant classes of time periodic perturbations

of a planar motion around an attractive center modeled by the potential (1). The method, a semi-analytical one, is based on the fact that we were able to obtain in a closed form the equation motion of the homoclinic orbit and perform a change of variable in the Melnikov function that maps the infinite integration interval into a finite one. This allows us to make predictions about the zeros of the Melnikov function — therefore about the onset of chaos — for significant physically motivated families of periodic perturbations.

Of course, one can always numerically compute the equivalent function  $K(\Omega)$  for any given function  $f(r)$  and any value of  $\Omega$  and  $\beta$  in the range  $1/16 \leq \beta \leq 1/12$ . Since  $\tau(r)$  is known explicitly, this computation can easily be made with a great precision. For the perturbations here considered,  $f(r)$  a power of  $r$ , and a fixed  $\Omega > \Omega_k$ , we have that  $K(\Omega)$  takes positive and negative values depending on the values of  $\beta$ . Then for certain specific values of  $\beta$  we will have  $K(\Omega) = 0$  and the Melnikov method does not apply in these cases. Therefore it does look like that the generic situation is chaotic, i.e., given a function  $f(r) \sim r^\alpha$  and a value of  $\beta$ , only for a numerable set of frequencies  $\Omega$  we should not have chaos. To be more precise, the Melnikov method fails to predict chaos only for a numerable set of perturbation frequencies  $\Omega$ . We note that seldom one can find so easily the parameters range of a given problem that will produce a chaotic situation. Therefore, based in our previous analysis, we can conjecture that chaos is generic for the class of perturbed systems studied in this communication.

Finally, we want to comment on the fully numerical approaches to the Melnikov method. In order to prove numerically the existence of crossings of the homoclinic orbits the analyticity of the Hamiltonian is required [8, 16]. But numerics can give only indications of the existence of derivatives to a finite order. Then the use of the Melnikov method in this case is not in conclusive.

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## FIGURE CAPTIONS

Fig.1. The effective potential is plotted for the dimensionless parameter:  $\beta = 0.068$  (top curve), 0.072, and 0.078 (bottom curve); the maximum value is located at  $r_{un} = (1 - \sqrt{1 - 12\beta})/2$ .

Fig.2. A graphic of the positive branch of Eq. (9) is pictured for values of the parameter  $\beta = 0.064$  (top curve), 0.065, 0.066, 0.067, and 0.068 (bottom curve).

Fig.3. A graphic of  $\sin[\Omega\tau(r)]$  for  $\beta = 0.064$ , and  $\Omega = 0.05$  (top curve), 0.06, 0.07, and 0.08 (bottom curve).

Fig.4. Plot of the integrand of  $K(\Omega)$  for  $\beta=0.064$ ,  $\Omega=0.06$ , and  $f'(r) \equiv df/dr = r$  (top curve),  $r^2/10$ , and  $r^3/100$  (bottom curve).

Fig.5. Plot of the integrand of  $K(\Omega)$  for  $\beta=0.068$ ,  $\Omega=0.15$ , and  $f'(r) = 1/(3r^2)$  (top curve),  $1/(9r^3)$ , and  $1/((27r^4)$ , (bottom curve).











